Final Exam (Back-up) ISI Bangalore January 2015

Algebra III 100 Points

Notes.
(a) Begin each answer on a separate sheet and ensure that the answers to all the parts to a question are arranged contiguously.
(b) Assume only those results that have been proved in class. All other steps should be justified.
(c) $\mathbb{Z}$ = integers, $\mathbb{Q}$ = rational numbers, $\mathbb{R}$ = real numbers $\mathbb{C}$ = complex numbers.

---

1. [10 points] Find an $n \times n$ matrix over $\mathbb{Q}$ whose characteristic polynomial is

$$\sum_{i=1}^{n+1} 2^i X^{n-i} = X^n + 2X^{n-1} + \cdots + 2^n.$$

2. [12 points] Let G be the abelian group generated by x, y, z satisfying the relations

$$x + y = 0, \quad 2x = 0, \quad 4x + 2z = 0, \quad 4x + 2y + 2z = 0.$$

Write G as a direct product of cyclic groups.

3. [20 points] Which of the following are algebraic integers ? (Give brief justifications for your answers.)

$$2^{1/3} + 1, \quad 2^{-1/3} + 1, \quad (\sqrt{-3} + 1)/2, \quad (\sqrt{-5} + 1)/2.$$

4. [16 points]
Prove that 5^{2n+1} can be written as a sum of two squares in $\mathbb{Z}$ in exactly $2n + 1$ ways. (Hint: Use factorization in the ring of Gaussian integers.)

5. [6 points] Factor $7 - i$ as a product of primes in $\mathbb{Z}[i]$.

6. [15 points] Find 3 monic polynomials of degree 3 in $\mathbb{Z}[X]$ whose corresponding images in $(\mathbb{Z}/3\mathbb{Z})[X]$ are distinct and irreducible. Explain why each of these polynomials is also irreducible over $\mathbb{Q}$.

7. [6 points] Prove that there are infinitely many units in the ring $\mathbb{Z}[\sqrt{2}]$. (Hint: $\sqrt{2} + 1$ is a unit.)

8. [15 points] Give an example of a maximal ideal $\mathfrak{m}$ in $\mathbb{R}[X, Y]$ such that $\mathfrak{m} \neq (X - a, Y - b)$ for any $a, b \in \mathbb{R}$.